Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

The Counter-Pedal Surface of the Ellipsoid.

By Thomas Craig, *Johns Hopkins University.*

The particular surface investigated in the present paper is merely one of a class whose properties do not seem to have been studied; there is at least no reference to the surfaces in any works that I am acquainted with; further, Prof. Cayley expresses his belief that the subject is quite new. The definition of a counter-pedal surface is obtained from the definition of a pedal surface by a very slight change of the wording, thus: a pedal surface is the locus of the intersection of the tangent planes to a given surface with straight lines drawn from a given point parallel to the normal; the counter-pedal surface is the locus of the intersection of the normals to a given surface with planes drawn through a given point parallel to the tangent planes. We can have in like manner pedal curves and counter-pedal curves. Problems connected with counter-pedals, whether curves or surfaces, will in general be more difficult than those connected with pedals, as it would seem that in general the degree of the counter-pedal is higher than that of the pedal. Before taking up the analytical part of the work I must first express my great obligations to Prof. Cayley for the assistance he has given me in this connection, especially in finding the equation of the surface. I had found the equation of the surface, but in a complicated form and encumbered with a factor of the sixth degree, which I knew must exist but could not find. On presenting my difficulty to Prof. Cayley, he was good enough to work out for me the equation of the surface in another and simpler form. The investigation below to find the equation of the surface is, with some insignificant changes, entirely due to Prof. Cayley. I have also received assistance from him on many other minor points in working out the properties of this surface. I have not as yet done much on the general theory of counter-pedal surfaces and curves, but if one may judge anything from analogy in the case of the counter-pedals of the ellipse and ellipsoid, these new curves and surfaces seem to be more closely related to negative pedals than to the direct pedals. For example, the degree of the negative and counter-pedals of the ellipse is six; the degree of the negative and counter-pedals of the ellipsoid is ten; while the degree of the pedals is in each case four. In both of these cases the pole is supposed to be at the centre. If for the ellipse the pole is taken at the focus, the negative pedal is a curve of the fourth degree having the lines $x^2+y^2=0$ for stationary tangents.

The counter-pedal surface of the ellipsoid, as considered in the following, is the locus of the intersections of central planes parallel to the tangent planes of the ellipsoid with the normals at the corresponding points of contact. The ellipsoid is given by the equation

$$\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}=1\,.$$

The length of the central perpendicular upon the tangent plane to the ellipsoid at any point may be denoted by P; then

$$\frac{1}{P^2}=\frac{\xi^2}{a^4}+\frac{\eta^2}{b^4}+\frac{\zeta^2}{c^4}\,;$$

further write

$$\alpha\,,\ \beta\,,\ \gamma=b^2-c^2\,,\ c^2-a^2\,,\ a^2-b^2\,.$$

Denote by x, y, z the coördinates of a point on the counter-pedal corresponding to a point ξ, η, ζ on the ellipsoid, then it is obvious that these are given by the equations

$$x=\xi\left(1-\frac{P^2}{a^2}\right),\quad y=\eta\left(1-\frac{P^2}{b^2}\right),\quad z=\zeta\left(1-\frac{P^2}{c^2}\right).$$

In order to find the equation of the counter-pedal we have to eliminate ξ, η, ζ between these equations and

$$\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}=1\,.$$

It is to be noticed that these equations give

$$x\xi+y\eta+z\zeta=x^2+y^2+z^2\,,$$

$$\frac{x\xi}{a^2}+\frac{y\eta}{b^2}+\frac{z\zeta}{c^2}=0\,,$$

$$\frac{\xi}{a^2}(y\zeta-z\eta)+\frac{\eta}{b^2}(z\xi-x\zeta)+\frac{\zeta}{c^2}(x\eta-y\xi)=0\,.$$

The geometrical meaning of these equations is obvious; one should be able to eliminate between these and the equation of the ellipsoid and find the equation of the surface. I was, however, not able to do it. The result was of the degree sixteen, instead of ten, the proper degree; but I could not find the required factor. Instead of $-P^2$ it will be convenient to write merely t, then

$$x = \xi\left(\frac{a^2+t}{a^2}\right), \quad y = \eta\left(\frac{b^2+t}{b^2}\right), \quad z = \zeta\left(\frac{c^2+t}{c^2}\right);$$

from these by aid of

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} = 1$$

we derive

$$\frac{a^2x^2}{(a^2+t)^2} + \frac{b^2y^2}{(b^2+t)^2} + \frac{c^2z^2}{(c^2+t)^2} = 1,$$

and

$$\frac{x^2}{(a^2+t)^2} + \frac{y^2}{(b^2+t)^2} + \frac{z^2}{(c^2+t)^2} = -\frac{1}{t}.$$

For greater convenience replace x^2, y^2, z^2, a^2, b^2, c^2 by x, y, z, a, b, c, then to find the required equation we have to eliminate t between

$$\frac{ax}{(a+t)^2} + \frac{by}{(b+t)^2} + \frac{cz}{(c+t)^2} = 1$$

$$\frac{x}{(a+t)^2} + \frac{y}{(b+t)^2} + \frac{z}{(c+t)^2} = -\frac{1}{t}.$$

Multiply the second of these by t and add to the first; this gives

$$\frac{x}{a+t} + \frac{y}{b+t} + \frac{z}{c+t} = 0,$$

which is to be combined with

$$\frac{ax}{(a+t)^2} + \frac{by}{(b+t)^2} + \frac{cz}{(c+t)^2} = 1.$$

We have thus to eliminate t between a sextic and a quadratic equation. The method followed by Prof. Cayley to effect this elimination is as follows: Expanding the quadratic it is

$$(x+y+z)t^2 + [(b+c)x + (c+a)y + (a+b)z]\,t + bcx + cay + abz = 0.$$

Write this as

$$Pt^2 + Qt + R = 0.$$

Denote by t_1 and t_2 the roots of this quadratic, then we have

$$Pt^2 + Qt + R = P(t-t_1)(t-t_2),$$

$$t_1 + t_2 = -\frac{Q}{P}, \quad t_1 t_2 = \frac{R}{P}.$$

The minimum value of t is of course $-a$, and its maximum value $-c$. Make $t=-a$, then we have

$$a^2(x+y+z)-a(\overline{b+c}\,x+\overline{c+a}\,y+\overline{a+b}\,z)+bcx+cay+abz=P(a+t_1)(a+t_2)$$

and obviously

$$P(a+t_1)(a+t_2)=-\beta\gamma x$$
$$P(b+t_1)(b+t_2)=-\gamma\alpha y$$
$$P(c+t_1)(c+t_2)=-\alpha\beta z$$

The quantity $(t_1-t_2)^2$ will be needed; this is given by

$$(t_1-t_2)^2=\frac{Q^2-4PR}{P^2}\,.$$

Substituting for P, Q, R their values, this is

$$(t_1-t_2)^2=\frac{\alpha^2x^2+\beta^2y^2+\gamma^2z^2-2\beta\gamma yz-2\alpha\beta xy-2\gamma\alpha zx}{(x+y+z)^2}$$

or say

$$(t_1-t_2)^2=\frac{V}{P^2}\,.$$

The values of t_1 and t_2, substituted in the sextic equation, will lead to the expression required for the equation of the counter-pedal. The sextic expanded is

$$(b+t)^2(c+t)^2ax+(c+t)^2(a+t)^2by+(b+t)^2(a+t)^2cz-(a+t)^2(b+t)^2(c+t)^2=0\,.$$

Write this as

$$Aax+Bby+Ccz-D=0\,;$$

denote by the suffix 1 or 2 the result of substituting for t its values t_1 and t_2; multiplying together the two identical equations thus obtained we have

$$(A_1ax+B_1by+C_1cz-D_1)(A_2ax+B_2by+C_2cz-D_2)=0\,;$$

this expanded is

$$A_1A_2a^2x^2+B_1B_2b^2y^2+C_1C_2c^2z^2+D_1D_2$$
$$+(B_1C_2+B_2C_1)\,bcyz+(A_1C_2+A_2C_1)\,cazx+(A_1B_2+A_2B_1)\,abxy$$
$$-(A_1D_2+A_2D_1)\,ax-(B_1D_2+B_2D_1)\,by-(C_1D_2+C_2D_1)\,cz=0\,.$$

Using the above equations giving the values of x, y, z, the coefficients in this are readily found.

$$A_1A_2=(b+t_1)^2(b+t_2)^2(c+t_1)^2(c+t_2)^2=\frac{\alpha^2\beta^2\gamma^2}{P^4}\cdot\alpha^2y^2z^2$$

$$D_1D_2=(a+t_1)^2(a+t_2)^2\,.\,.\,.\,.\,.\,(c+t_2)^2=\frac{\alpha^2\beta^2\gamma^2}{P^6}\cdot\alpha^2\beta^2\gamma^2x^2y^2z^2$$

$$\begin{aligned}
B_1C_2+B_2C_1 &= (c+t_1)^2(a+t_1)^2(a+t_2)^2(b+t_2)^2+(c+t_2)^2(a+t_2)^2(a+t_1)^2(b+t_1)^2 \\
&= (a+t_1)^2(a+t_2)^2\left[(c+t_1)^2(b+t_2)^2+(c+t_2)^2(b+t_1)^2\right] \\
&= (a+t_1)^2(a+t_2)^2\left[\alpha^2(t_1-t_2)^2+2\,\overline{b+t_1}\,.\,\overline{b+t_2}\,.\,\overline{c+t_1}\,.\,\overline{c+t_2}\right] \\
&= \frac{\beta^2\gamma^2x^2}{P^2}\left[\frac{\alpha^2\nabla}{P^2}+2\frac{\alpha^2\beta\gamma yz}{P^2}\right]
\end{aligned}$$

$$\begin{aligned}
A_1D_2+A_2D_1 &= (b+t_1)^2(c+t_1)^2(a+t_2)^2(b+t_2)^2(c+t_2)^2 \\
&\qquad +(b+t_2)^2(c+t_2)^2(a+t_1)^2(b+t_1)^2(c+t_1)^2 \\
&= (b+t_1)^2(b+t_2)^2(c+t_1)^2(c+t_2)^2\left[(a+t_2)^2+(a+t_1)^2\right] \\
&= \frac{\alpha^2\beta^2\gamma^2}{P^4}\cdot\alpha^2y^2z^2\left[(t_1-t_2)^2+2(a+t_1)(a+t_2)\right] \\
&= \frac{\alpha^2\beta^2\gamma^2}{P^4}\cdot\alpha^2y^2z^2\left[\frac{\nabla}{P^2}-\frac{2\beta\gamma x}{P}\right]
\end{aligned}$$

The remaining values are written down by symmetry. Substitution of these values in the above equation gives

$$\begin{gathered}
\frac{\alpha^2\beta^2\gamma^2}{P^4}x^2y^2z^2\left[a^2\alpha^2+b^2\beta^2+c^2\gamma^2\right]+\frac{\alpha^2\beta^2\gamma^2}{P^6}x^2y^2z^2 \\
+xyz\frac{\alpha^2\beta^2\gamma^2}{P^4}\left\{(bcx+cay+abz)\nabla+2xyz\left[bc\beta\gamma+ca\gamma\alpha+ab\alpha\beta\right]\right\} \\
-xyz\frac{\alpha^2\beta^2\gamma^2}{P^6}\left\{\nabla(a\alpha^2yz+b\beta^2zx+c\gamma^2xy)-2xyz\,.\,\alpha\beta\gamma P\left[a\alpha+b\beta+c\gamma\right]\right\}=0\,.
\end{gathered}$$

The whole expression divides by the factor

$$xyz\,.\,\frac{\alpha^2\beta^2\gamma^2}{P^4}\,;$$

effect this division, then multiply out by P^2 and reduce further by noting that $a\alpha+b\beta+c\gamma=0$. The terms in the result containing P^2xyz as a factor are easily seen to be

$$P^2xyz\left[a^2\alpha^2+b^2\beta^2+c^2\gamma^2+2\beta\gamma bc+2\gamma\alpha ca+2\alpha\beta ab\right]=P^2xyz(a\alpha+b\beta+c\gamma)^2=0.$$

The terms remaining are now

$$\alpha^2\beta^2\gamma^2xyz-(a\alpha^2yz+b\beta^2zx+c\gamma^2xy)\nabla+P^2(bcx+cay+abz)\nabla=0\,.$$

This is the equation of a skew surface of the fifth degree. The principal section in the plane of xy is the cubic

$$(a-b)^2xy-(bx+ay)(x+y)^2=0\,.$$

This has an asymptote

$$bx+ay+ab=0$$

which meets the curve in

$$ax+by=0\,.$$

There is further an asymptotic parabola

$$(x+y)^2 = ax + by + ab\,,$$

and this meets the curve in

$$x + y = -\frac{ab}{a+b}$$

$$bx + ay = \left(\frac{ab}{a+b}\right)^2.$$

The discussion of this surface is reserved for the present, but we may just notice the values of the coördinates of a point on any of the principal sections corresponding to a given value of t. We have

$$\frac{ax}{(a+t)^2} + \frac{by}{(b+t)^2} + \frac{cz}{(c+t)^2} = 1\,,$$

$$\frac{x}{a+t} + \frac{y}{b+t} + \frac{c}{c+t} = 0\,.$$

For the section in the plane $x = 0$ we may write

$$y = M(b+t)\,, \qquad z = -M(c+t)\,,$$

then

$$M\left(\frac{b}{b+t} - \frac{c}{c+t}\right) = 1\,,$$

and so

$$M = \frac{(b+t)(c+t)}{(b-c)\,t}\,.$$

Introducing this value of M we have the values of y and z corresponding to $x = 0$. Similarly the coördinates of a point in the other two planes may be given. Arranged in tabular form these are

$x = 0$	$x = -\frac{(c+t)(a+t)^2}{(c-a)\,t}$	$x = \frac{(a+t)^2(b+t)}{(a-b)\,t}$
$y = \frac{(b+t)^2(c+t)}{(b-c)\,t}$	$y = 0$	$y = -\frac{(a+t)(b+t)^2}{(a-b)\,t}$
$z = -\frac{(b+t)(c+t)^2}{(b-c)\,t}$	$z = \frac{(c+t)^2(a+t)}{(c-a)\,t}$	$z = 0$

The equations of the sections of the skew surface corresponding to these sets of values of x, y, z are

$$(b-c)^2yz - (cy+bz)(y+z)^2 = 0\,,$$

$$(c-a)^2zx - (az+cx)(z+x)^2 = 0\,,$$

$$(a-b)^2xy - (bx+ay)(x+y)^2 = 0\,.$$

If in the equation of the skew surface we replace x, y, z, a, b, c by the squares of these quantities we have at once the required equation of the counter-pedal. The correspondence between these two surfaces is such that to every generator on the ruled surface corresponds a certain curve on the counter-pedal. The curve and generator are found by giving t a certain constant value in both cases. The lines on the ellipsoid corresponding to a given value of t are readily found. Required to find the line on

$$\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}=1$$

for which t, *i. e.* $-P^2$, has a constant value, say $-t_0$: this gives

$$\frac{\xi^2}{a^4}+\frac{\eta^2}{b^4}+\frac{\zeta^2}{c^4}=\frac{1}{t_0}\,;$$

the required curve is then the intersection of the two ellipsoids

$$\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}=1$$

$$\frac{t_0\xi^2}{a^4}+\frac{t_0\eta^2}{b^4}+\frac{t_0\zeta^2}{c^4}=1\,.$$

The projections of this curve on the coördinate planes are evidently conics; the projection on the plane of xz for example is the hyperbola

$$\frac{\xi^2}{a^2}\cdot\frac{a^2-b^2}{a^2}-\frac{\zeta^2}{c^2}\cdot\frac{b^2-c^2}{c^2}=\frac{t_0-b^2}{t_0}\,.$$

From their definition, these curves are obviously Poinsot's pollhodes. For the value $t_0=b^2$ the curve on the ellipsoid is the *separating* pollhode; its projection on the plane zx is the pair of lines

$$\frac{\xi^2}{a^2}\cdot\frac{a^2-b^2}{a^2}-\frac{\zeta^2}{c^2}\cdot\frac{b^2-c^2}{c^2}=0\,.$$

The pollhode itself consists of two ellipses whose planes pass through the axis of y. The curves on the counter-pedal corresponding to a certain constant value of t, *i. e.* $-P^2$, are found by laying off on the normals to the ellipsoid along the corresponding pollhode the length P. The extremities of these lines also lie in that parallel to the ellipsoid for which the modulus $=P$; the lines on the counter-pedal are then the intersections of the inner sheet of the parallel surface with the counter-pedal. These are only partial intersections, and I have not been able to determine the degree of the curve.

The equation of the counter-pedal is obtained from that of the skew surface as above indicated, and is

$$\begin{aligned}
&\alpha^2\beta^2\gamma^2x^2y^2z^2\\
&-(a^2\alpha^2y^2z^2+b^2\beta^2z^2x^2+c^2\gamma^2x^2y^2)(\alpha^2x^4+\beta^2y^4+\gamma^2z^4-2\alpha\beta xy-2\beta\gamma yz-2\gamma\alpha zx)\\
&+(x^2+y^2+z^2)^2(b^2c^2x^2+c^2a^2y^2+a^2b^2z^2)(\alpha^2x^4+\beta^2y^4+\gamma^2z^4\\
&\qquad\qquad-2\beta\gamma yz-2\gamma\alpha zx-2\alpha\beta xy)=0\,.
\end{aligned}$$

The counter-pedal is therefore of the tenth degree—the same degree as that of the negative pedal of the ellipsoid. The terms are grouped into those of the sixth, eighth, and tenth degrees respectively. Writing

$$A = [\Sigma x^2]^2\,,\ B = \Sigma\alpha^2 x^4\,,\ C = \Sigma b^2 c^2 x^2\,,$$

$$D = \Sigma a^2 \alpha^2 y^2 z^2\,,\ E = 2\Sigma\alpha\beta x^2 y^2\,,\ G = \alpha^2\beta^2\gamma^2 x^2 y^2 z^2\,,$$

the equation may be written

$$G + (B - E)(AC - D) = 0\,,$$

or, writing

$$B - E = -L\,, \qquad AC - D = M\,,$$

the equation is

$$G - LM = 0\,.$$

The L is the ∇ of the above. For $G = 0$ with $M = 0$ we have the principal sections of the surface—complete in the planes $z = 0$ and $x = 0$, but as we shall see not complete in $y = 0$. These principal sections are the counter-pedals of the principal sections of the ellipsoid together with, in the plane $y = 0$, a certain pair of straight lines. The equations of the counter-pedals of the principal ellipses are

$$(b^2 - c^2)^2 y^2 z^2 - (c^2 y^2 + b^2 z^2)(y^2 + z^2)^2 = 0\,,$$

$$(c^2 - a^2)^2 z^2 x^2 - (a^2 z^2 + c^2 x^2)(z^2 + x^2)^2 = 0\,,$$

$$(a^2 - b^2)^2 x^2 y^2 - (b^2 x^2 + a^2 y^2)(x^2 + y^2)^2 = 0\,.$$

The coördinates of a point on each of these sextic curves are given by the table

$x^2 = 0$	$x^2 = -\dfrac{(c^2 + t)(a^2 + t)^2}{(c^2 - a^2)t}$	$x^2 = \dfrac{(a^2 + t)^2(b^2 + t)}{(a^2 - b^2)t}$
$y^2 = \dfrac{(b^2 + t)^2(c^2 + t)}{(b^2 - c^2)t}$	$y^2 = 0$	$y = -\dfrac{(a^2 + t)(b^2 + t)^2}{(a^2 - b^2)t}$
$z^2 = -\dfrac{(b^2 + t)(c^2 + t)^2}{(b^2 - c^2)t}$	$z^2 = \dfrac{(c^2 + t)^2(a^2 + t)}{(c^2 - a^2)t}$	$z = 0$

It is important to notice that these values go in pairs. For values of $-t < b^2$ the first and second columns give real values of x, y and z, while the third gives imaginary values; for $-t = b^2$ the first and third columns vanish; for values of $-t > b^2$ the first column is imaginary while the second and third are real. This remark is important in making a drawing or model of the surface, as it shows just what points in the three cross sections are to be united.

If in the equation of the surface we make

$$G = 0 \text{ and } L = 0$$

we have for the three possible cases a pair of straight lines in each of the planes. These are

$$(\alpha x^2 - \beta y^2)^2 = 0 \quad \text{in } z = 0\,,$$
$$(\beta y^2 - \gamma z^2)^2 = 0 \quad \text{in } x = 0\,,$$
$$(\gamma z^2 - \alpha x^2)^2 = 0 \quad \text{in } y = 0\,.$$

Of these only the last,

$$\alpha x^2 - \gamma z^2 = 0\,,$$

is real and completes the principal section in the plane $y = 0$. The principal sections of the surface shown in isometric projections are given in figure 1.

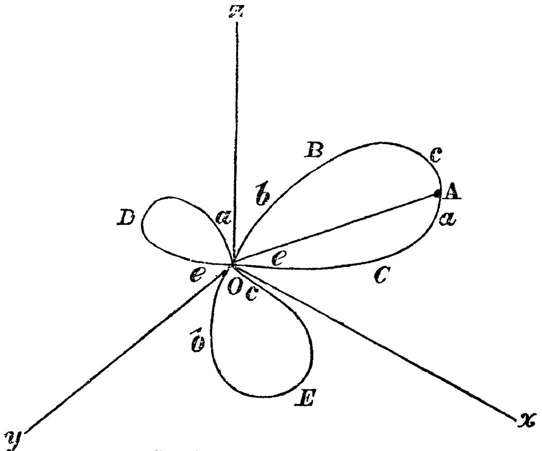

Fig.1.

The portion OBA of the loop in the plane $y = 0$ corresponds to values of t between $-b^2$ and $-c^2$ and is joined to the corresponding points in the loop E; the values of the corresponding points in OBA and E are found from the above table. The portion OCA is joined in like manner to D. The small letters indicate corresponding points and serve to show completely how a drawing or model of the surface may be constructed.

In making a drawing of the surface it will of course be desirable to work out the numerical values of a few of the corresponding points though the lines joining them will have to be filled in by the eye—unless indeed one chooses to go through a very tedious computation to find the intermediate points through which to trace them.

The lines

$$\alpha x^2 - \gamma z^2 = 0$$

are clearly double lines on the surface; a section of the surface through the point a of Fig. 1 will consist of two curves like Fig. 2.

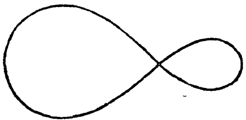

Fig. 2.

This section gives only eight real points of intersection with a right line. There are four real singular points of the nature of triple points in the plane of $y = 0$ and an equal number of imaginary ones in the other two planes; the origin is a singular point of a very high order and will hardly repay study. The double lines

$$\alpha x^2 - \gamma z^2 = 0$$

on the counter-pedal correspond to the separating pollhode on the ellipsoid; the projection of these lines has been shown to be

$$\frac{x^2}{a^2} \cdot \frac{\gamma}{a^2} - \frac{z^2}{c^2} \cdot \frac{\alpha}{c^2} = 0\,;$$

the angle between these projections and the double lines being called θ, we have

$$\cos\theta = (a^2 + c^2)\sqrt{\alpha\gamma} \div \sqrt{\alpha\gamma(a^4 + c^4) + a^4\alpha^2 + c^4\gamma^2}.$$

This is a right angle only for ellipsoids of revolution, as we can only have $\cos\theta = 0$ for $\alpha = 0$ or $\gamma = 0$. The projection and the double line will coincide with each other for $\cos\theta = 1$; this gives

$$b^2 = c^2\,\frac{a^2 - c^2}{a^2 + c^2}.$$

The double lines are at right angles for an ellipsoid whose axes are connected by the relation

$$b^2 = \frac{2a^2c^2}{a^2 + c^2}.$$

I thought that the umbilics of the ellipsoid might correspond to the above mentioned triple points or the counter-pedal; on testing I found that they did not correspond, though there is a singular relation between the coördinates of the points corresponding to the umbilics, and the coördinates of the triple points.

The coördinates of the umbilics on the ellipsoid are

$$\xi^2 = -a^2\frac{\gamma}{\beta}, \quad \eta = 0, \quad \zeta^2 = -c^2\frac{\alpha}{\beta};$$

the value of t for these points is $-\frac{a^2c^2}{b^2}$; substituting this in the second column of the above table we have

$$x^2 = -\frac{\gamma a^2}{b^2\beta}, \quad y = 0, \quad z^2 = -\frac{\gamma^2 a}{b^2\beta}.$$

The coördinates of the triple points on the counter-pedal are found by making $t = -b^2$; they are

$$x^2 = -\frac{\gamma^2 a}{b^2\beta}, \quad y = 0, \quad z^2 = -\frac{\gamma a^2}{b^2\beta}.$$

The values of x and z for the two series of points are just interchanged. The condition that the umbilics on the ellipsoid should correspond to the triple points on the counter-pedal is

$$b^2 = \frac{a^2 + c^2}{2}.$$

Besides the two straight double lines on the surface there are two other double lines whose projections are of a quadrifoil shape, the curves in space having the same shape only bent to fit the surface.

The existence of these lines is readily inferred from Fig. 1; they are produced by the intersection of the portion of the surface joining the loop E to the partial loop OBA with the portion which joins the loop D to the partial loop OCA. The degree of these nodal lines I have not been able to determine.

The lines of curvature on the counter-pedal to the ellipsoid are its intersections with the counter-pedals to the two confocal hyperholoids. I have not been able to deduce any results of consequence concerning the curvature of this surface, but in the next few pages I have given the values of all of the quantities employed by Gauss in his investigations on the curvature of surfaces. These values will certainly be necessary in any future study of the curvature of the counter-pedal, so it is worth while giving them here, although I have not been able to determine in a simple manner the radii of curvature.

Denoting by u and v the curvilinear coördinates of a point on the surface of the ellipsoid, we have

$$-\beta\gamma\xi^2 = a^2(a^2 + u)(a^2 + v),$$
$$-\gamma\alpha\eta^2 = b^2(b^2 + u)(b^2 + v),$$
$$-\alpha\beta\zeta^2 = c^2(c^2 + u)(c^2 + v).$$

Let P, P_1, P_2 denote the three perpendiculars from the centre upon the tangent planes to the three confocal quadrics at their common point ξ, η, ζ, then

$$\frac{1}{P^2}=\frac{\xi^2}{a^4}+\frac{\eta^2}{b^4}+\frac{\zeta^2}{c^4},$$

$$\frac{1}{P_1^2}=\frac{\xi^2}{(a^2+u)^2}+\frac{\eta^2}{(b^2+u)^2}+\frac{\zeta^2}{(c^2+u)^2},$$

$$\frac{1}{P_2^2}=\frac{\xi^2}{(a^2+v)^2}+\frac{\eta^2}{(b^2+v)^2}+\frac{\zeta^2}{(c^2+v)^2};$$

or in terms of u and v these are

$$P^2=\frac{a^2b^2c^2}{uv},$$

$$P_1^2=\frac{(a^2+u)(b^2+u)(c^2+u)}{u(u-v)},$$

$$P_2^2=\frac{(a^2+v)(b^2+v)(c^2+v)}{v(v-u)}.$$

The element of length ds given in the form

$$\sqrt{Edu^2+2Fdudv+Gdv^2}$$

is of course for the ellipsoid

$$=\frac{1}{2}\sqrt{\frac{du^2}{P_1^2}+\frac{dv^2}{P_2^2}}.$$

Denote by $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$ the quantities which on the counter-pedal correspond to E, F, G on the original surface:
then

$$\mathfrak{E}=\left(\frac{dx}{du}\right)^2+\left(\frac{dy}{du}\right)^2+\left(\frac{dz}{du}\right)^2$$

$$\mathfrak{F}=\frac{dx}{du}\,\frac{dx}{dv}+\frac{dy}{du}\,\frac{dy}{dv}+\frac{dz}{du}\,\frac{dz}{dv}$$

$$\mathfrak{G}=\left(\frac{dx}{dv}\right)^2+\left(\frac{dy}{dv}\right)^2+\left(\frac{dz}{dv}\right)^2.$$

Now

$$x=\xi\left(1-\frac{P^2}{a^2}\right)\ \&\text{c.}$$

and therefore

$$\frac{dx}{du}=\frac{d\xi}{du}-\frac{P^2}{a^2}\,\frac{d\xi}{du}-\frac{\xi}{a^2}\,\frac{dP^2}{du}\ \&\text{c.}$$

Substituting in the expression for $\mathfrak{E}$ we have

$$\mathfrak{E} = \Sigma\left(\frac{d\xi}{du}\right)^2 - 2P^2\Sigma\frac{1}{a^2}\left(\frac{d\xi}{du}\right)^2 - 2\frac{dP^2}{du}\Sigma\frac{\xi}{a^2}\frac{d\xi}{du}$$

$$+ 2P^2\frac{dP^2}{du}\Sigma\frac{\xi}{a^4}\frac{d\xi}{du} + P^4\Sigma\frac{1}{a^4}\left(\frac{d\xi}{du}\right)^2 + \left(\frac{dP^2}{du}\right)^2\Sigma\frac{\xi^2}{a^4}\,.$$

To reduce this, observe that

$$\frac{d\xi}{du} = \frac{1}{2}\frac{\xi}{a^2+u}\,,\ \text{\&c.}$$

$$\Sigma\frac{\xi}{a^2}\frac{d\xi}{du} = 0\,,$$

$$\Sigma\frac{\xi^2}{a^2(a^2+u)^2} = -\frac{1}{uP_1^2}\,, \qquad \Sigma\frac{\xi^2}{a^4(a^2+u)} = \frac{1}{uP^2}\,,$$

$$\Sigma\frac{\xi^2}{a^4(a^2+u)^2} = \frac{1}{u^2}\frac{P^2+P_1^2}{P^2P_1^2}\,, \qquad \frac{dP^2}{du} = -\frac{P^2}{u}\,, \qquad \left(\frac{dP}{du}\right)^2 = \frac{1}{4}\frac{P^2}{u^2}\,.$$

Making these and other obvious reductions we have

$$\mathfrak{E} = E + \frac{P^2}{2uP_1^2} + \frac{P^2(P^2+P_1^2)}{4uP_1^2}$$

or

$$\mathfrak{E} = E + \frac{P^2}{4u^2P_1^2}\,[P^2 + 2u + P_1^2]\,;$$

of course it follows from the symmetry of $\mathfrak{E}$ and $\mathfrak{G}$ that

$$\mathfrak{G} = G + \frac{P^2}{4v^2P_2^2}\,[P^2 + 2v + P_2^2]\,.$$

The reductions to be performed in order to find $\mathfrak{F}$ are a little different from the above, but there is no difficulty in getting it; all three of these quantities may be written as follows:

$$\mathfrak{E} = \frac{P^2}{4P_1^2}\left(\frac{1}{P} + \frac{P}{u}\right)^2 + \frac{1}{4}\frac{P^2}{u^2}\,,$$

$$\mathfrak{G} = \frac{P^2}{4P_2^2}\left(\frac{1}{P} + \frac{P}{v}\right)^2 + \frac{1}{4}\frac{P^2}{v^2}\,,$$

$$\mathfrak{F} = \frac{1}{4}\frac{P^4}{a^2b^2c^2}\,, = \frac{1}{4}\frac{P^2}{uv}\,.$$

The element of length on the counter-pedal being now given by

$$d\sigma^2 = \mathfrak{E}\,du^2 + 2\mathfrak{F}\,du\,dv + \mathfrak{G}\,dv^2$$

becomes

$$d\sigma^2 = \frac{P^2}{4}\left\{\left[\left(\frac{1}{P} + \frac{P}{u}\right)\frac{du}{P_1}\right]^2 + \left[\left(\frac{1}{P} + \frac{P}{v}\right)\frac{dv}{P_2}\right]^2 + \left(\frac{du}{u} + \frac{dv}{v}\right)^2\right\}.$$

The element of area of the counter-pedal is

$$d\Sigma = \sqrt{\mathfrak{E}\mathfrak{G}-\mathfrak{F}^2}\,dudv\,, \;=Ududv\,.$$

Now

$$U^2=\frac{P^4}{16P_1^2P_2^2}\left\{\left(\frac{1}{P}+\frac{P}{u}\right)^2\left(\frac{1}{P}+\frac{P}{v}\right)^2+\frac{P_2^2}{v^2}\left(\frac{1}{P}+\frac{P}{u}\right)^2+\frac{P_1^2}{u^2}\left(\frac{1}{P}+\frac{P}{v}\right)^2\right\}.$$

This is easily brought into the form

$$U^2=\frac{1}{16}\,\frac{(P^2+u)^2(P^2+v)^2}{P_1^2P_2^2u^2v^2}\left[1+\frac{P^2P_1^2}{(P^2+u)^2}+\frac{P^2P_2^2}{(P^2+v)^2}\right],$$

or again

$$U^2=\frac{1}{16}\,\frac{1}{P_1^2P_2^2}\left[\frac{(P^2+u)^2(P^2+v)^2}{u^2v^2}+\frac{P^2P_1^2\,(P^2+v)^2}{u^2v^2}+\frac{P^2P_2^2(P^2+u)^2}{u^2v^2}\right],$$

or say

$$U^2=\frac{1}{16}\,\frac{T^2}{P_1^2P_2^2}\,;$$

the element of area is now

$$d\Sigma=\frac{1}{4}\,\frac{T}{P_1P_2}\,dudv\,,$$

and for the total area we must have

$$\Sigma=2\int_{-a^2}^{-b^2}\int_{-b^2}^{-c^2}\frac{T}{P_1P_2}\,dudv:$$

of course the area of the ellipsoid is

$$S=2\int_{-a^2}^{-b^2}\int_{-b^2}^{-c^2}\frac{dudv}{P_1P_2}\,.$$

The ratio between corresponding elements of area on the ellipsoid and its counter-pedal is

$$\frac{d\Sigma}{dS}=T.$$

Denoting by $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ the direction-cosines of the normals to the ellipsoid and counter-pedal respectively at corresponding points, we have

$$T=\begin{vmatrix} \alpha_2\,, & \dfrac{dx}{d\xi}\,, & \dfrac{dx}{d\eta}\,, & \dfrac{dx}{d\zeta} \\ \beta_2\,, & \dfrac{dy}{d\xi}\,, & \dfrac{dy}{d\eta}\,, & \dfrac{dy}{d\zeta} \\ \gamma_2\,, & \dfrac{dz}{d\xi}\,, & \dfrac{dz}{d\eta}\,, & \dfrac{dz}{d\zeta} \\ 0 & \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix}.$$

The verification of this value of T by working out the determinant would be a very difficult matter, as the values of α_2, β_2, γ_2, whether obtained directly from the equation of the surface or in the manner indicated below, are extremely complicated.

Using the notation employed by Salmon, *Geom. of three Dimensions*, write

$$A,\ B,\ C = \frac{d(\eta,\zeta)}{d(u,v)},\ \frac{d(\zeta,\xi)}{d(u,v)},\ \frac{d(\xi,\eta)}{d(u,v)}$$

referring to the ellipsoid; also

$$\text{Ɐ},\ \text{ꓭ},\ \text{Ɔ} = \frac{d(y,z)}{d(u,v)},\ \frac{d(z,x)}{d(u,v)},\ \frac{d(x,y)}{d(u,v)}$$

referring to the counter-pedal. For convenience I will write the coördinates x, y, z as

$$x = \xi\left(1+\frac{k}{a^2}\right) \text{ \&c., } i.\ e.\ k = -P^2;$$

also write

$$\bar{a},\ \bar{b},\ \bar{c} = \left(1+\frac{k}{a^2}\right),\ \left(1+\frac{k}{b^2}\right),\ \left(1+\frac{k}{c^2}\right).$$

We find now at once

$$\text{Ɐ} = \overline{bc}\,\frac{d(\eta,\zeta)}{d(u,v)} + \frac{\bar{c}\eta}{b^2}\,\frac{d(k,\zeta)}{d(u,v)} + \frac{\bar{b}\zeta}{c^2}\,\frac{d(\eta,k)}{d(u,v)}.$$

or

$$\text{Ɐ} = -\begin{vmatrix} -\overline{bc}, & \dfrac{\bar{c}\eta}{b^2}, & \dfrac{\bar{b}\zeta}{c^2} \\ \dfrac{dk}{du}, & \dfrac{d\eta}{du}, & \dfrac{d\zeta}{du} \\ \dfrac{dk}{dv}, & \dfrac{d\eta}{dv}, & \dfrac{d\zeta}{dv} \end{vmatrix}$$

$$= A - \begin{vmatrix} -\left(\dfrac{k}{b^2}+\dfrac{k}{c^2}+\dfrac{k^2}{b^2c^2}\right), & \dfrac{\eta}{b^2}\left(1+\dfrac{k}{c^2}\right), & \dfrac{\zeta}{c^2}\left(1+\dfrac{k}{b^2}\right) \\ \dfrac{dk}{du}, & \dfrac{d\eta}{du}, & \dfrac{d\zeta}{du}, \\ \dfrac{dk}{dv}, & \dfrac{d\eta}{dv}, & \dfrac{d\zeta}{dv}, \end{vmatrix}$$

or finally

$$V = A - \begin{vmatrix} \frac{-k}{b^2c^2}\left(b^2+c^2\right), & \frac{\eta}{b^2}, & \frac{\zeta}{c^2} \\ \frac{dk}{du}, & \frac{d\eta}{du}, & \frac{d\zeta}{du} \\ \frac{dk}{dv}, & \frac{d\eta}{dv}, & \frac{d\zeta}{dv} \end{vmatrix} - \frac{k}{b^2c^2} \begin{vmatrix} -k, & \eta, & \zeta \\ \frac{dk}{du}, & \frac{d\eta}{du}, & \frac{d\zeta}{du} \\ \frac{dk}{dv}, & \frac{d\eta}{dv}, & \frac{d\zeta}{dv} \end{vmatrix}.$$

The values of $\rotatebox{180}{$B$}$ and $\rotatebox{180}{$C$}$ may of course be written down at once by symmetry.

There are three other quantities necessary in obtaining the expressions for the radii of curvature, &c., by Gauss's method, viz.

$$\rotatebox{180}{E}' = \begin{vmatrix} \frac{d^2x}{du^2}, & \frac{d^2y}{du^2}, & \frac{d^2z}{du^2} \\ \frac{dx}{du}, & \frac{dy}{du}, & \frac{dz}{du} \\ \frac{dx}{dv}, & \frac{dy}{dv}, & \frac{dz}{dv} \end{vmatrix} \text{\&c.;}$$

or

$$\rotatebox{180}{E}' = \frac{d^2x}{du^2}\rotatebox{180}{A} + \frac{d^2y}{du^2}\rotatebox{180}{B} + \frac{d^2z}{du^2}\rotatebox{180}{C}$$

$$\rotatebox{180}{F}' = \frac{d^2x}{dudv}\rotatebox{180}{A} + \frac{d^2y}{dudv}\rotatebox{180}{B} + \frac{d^2z}{dudv}\rotatebox{180}{C}$$

$$\rotatebox{180}{G}' = \frac{d^2x}{dv^2}\rotatebox{180}{A} + \frac{d^2y}{dv^2}\rotatebox{180}{B} + \frac{d^2z}{dv^2}\rotatebox{180}{C}.$$

The values of these expressed as determinants of the fourth order are easily found; they are

$$\rotatebox{180}{E}' = \begin{vmatrix} \overline{abc}, & \frac{-\overline{bc}}{a^2}\xi, & \frac{-\overline{ca}}{b^2}\eta, & \frac{-\overline{ab}}{c^2}\zeta \\ \frac{d^2k}{du^2}, & \frac{d^2\xi}{du^2}, & \frac{d^2\eta}{du^2}, & \frac{d^2\zeta}{du^2} \\ \frac{dk}{du}, & \frac{d\xi}{du}, & \frac{d\eta}{du}, & \frac{d\zeta}{du} \\ \frac{dk}{dv}, & \frac{d\xi}{dv}, & \frac{d\eta}{dv}, & \frac{d\zeta}{dv} \end{vmatrix} + 2\frac{dk}{du}\left\{\frac{1}{a^2}\frac{d\xi}{du}\rotatebox{180}{A} + \frac{1}{b^2}\frac{d\eta}{du}\rotatebox{180}{B} + \frac{1}{c^2}\frac{d\zeta}{du}\rotatebox{180}{C}\right\}.$$

Writing $\overline{abc} = T + 1$, this becomes

$$\mathfrak{E}' = E' + 2\,\frac{dk}{du}\Sigma\frac{1}{a^2}\frac{d\xi}{du}V - \begin{vmatrix} -T, & \frac{\overline{bc}}{a^2}\xi, & \frac{\overline{ca}}{b^2}\eta, & \frac{\overline{ab}}{c^2}\zeta \\ \frac{d^2k}{du^2}, & \frac{d^2\xi}{du^2}, & \frac{d^2\eta}{du^2}, & \frac{d^2\zeta}{du^2} \\ \frac{dk}{du}, & \frac{d\xi}{du}, & \frac{d\eta}{du}, & \frac{d\zeta}{du} \\ \frac{dk}{dv}, & \frac{d\xi}{dv}, & \frac{d\eta}{dv}, & \frac{d\zeta}{dv} \end{vmatrix}.$$

The expression for $\mathfrak{D}'$ may be written down by symmetry; that for $\mathfrak{F}'$ is

$$\mathfrak{F}' = \frac{dk}{dv}\Sigma\frac{1}{a^2}\frac{d\xi}{du}V + \frac{dk}{du}\Sigma\frac{1}{a^2}\frac{d\xi}{dv}V - \begin{vmatrix} -T, & \frac{\overline{bc}}{a^2}\xi, & \frac{\overline{ca}}{b^2}\eta, & \frac{\overline{ab}}{c^2}\zeta \\ \frac{d^2k}{dudv}, & \frac{d^2\xi}{dudv}, & \frac{d^2\eta}{dudv}, & \frac{d^2\zeta}{dudv} \\ \frac{dk}{du}, & \frac{d\xi}{du}, & \frac{d\eta}{du}, & \frac{d\zeta}{du} \\ \frac{dk}{dv}, & \frac{d\xi}{dv}, & \frac{d\eta}{dv}, & \frac{d\zeta}{dv} \end{vmatrix}.$$

I have not been able to simplify these expressions any considerable amount.

If we denote by $\overline{x}, \overline{y}, \overline{z}$ the coördinates of a point on the *pedal* of the ellipsoid, it is easy to see that

$$\xi = x + \overline{x}, \quad \eta = y + \overline{y}, \quad \zeta = z + \overline{z}$$

or

$$x = \xi - \overline{x}, \quad y = \eta - \overline{y}, \quad z = \zeta - \overline{z}.$$

If $\overline{E}, \overline{F}, \overline{G}$ refer to the pedal, we have for the element of length on that surface

$$d\overline{\sigma}^2 = \overline{E}du^2 + 2\overline{F}dudv + \overline{G}dv^2.$$

It is quite easy to show that

$$\mathfrak{E} = E + \overline{E} + \frac{P^2}{2uP_1^2},$$

$$\mathfrak{D} = G + \overline{G} + \frac{P^2}{2vP_2^2},$$

$$\mathfrak{F} = \overline{F},$$

and then for the element of length on the counter-pedal

$$d\sigma^2 = ds^2 + d\bar{\sigma}^2 + \frac{P^2}{2}\left(\frac{du^2}{uP_1^2} + \frac{dv^2}{vP_2^2}\right);$$

writing

$$\Lambda^2 = a^2b^2c^2\,,$$
$$\Lambda_1^2 = a^2 + u\,.\,b^2 + u\,.\,c^2 + u\,,$$
$$\Lambda_2^2 = a^2 + v\,.\,b^2 + v\,.\,c^2 + v\,,$$

the last term in $d\sigma^2$ becomes

$$\frac{P^2}{2}(u - v)\left[\frac{du^2}{A_1^2} - \frac{dv^2}{A_2^2}\right],$$

$$= \frac{A^2}{2}\left(\frac{1}{v} - \frac{1}{u}\right)\left[\frac{du^2}{A_1^2} - \frac{dv^2}{A_2^2}\right];$$

and so

$$d\sigma^2 = ds^2 + d\bar{\sigma}^2 + \frac{A^2}{2}\left(\frac{1}{v} - \frac{1}{u}\right)\left[\frac{du^2}{A_1^2} - \frac{dv^2}{A_2^2}\right].$$

It would be interesting to find an interpretation of this last term, but I see no means of doing it.

Denoting by $d\bar{\Sigma}$ the element of area on the pedal, we have

$$d\bar{\Sigma} = \sqrt{\bar{E}\bar{G} - \bar{F}^2}\,dudv\,;$$

also

$$\bar{E} = \frac{P^2}{4u^2}\cdot\frac{P^2 + P_1^2}{P_1^2}\,,$$

$$\bar{G} = \frac{P^2}{4v^2}\cdot\frac{P^2 + P_2^2}{P_2^2}\,,$$

$$\bar{F} = \frac{1}{4}\cdot\frac{P^2}{uv}\,;$$

calling the quantity under the radical sign $\bar{V}^2$ we have

$$\bar{V}^2 = \frac{P^6}{16u^2v^2P_1^2P^2}\left[P^2 + P_1^2 + P_2^2\right],$$

and for the ratio between the elements of area of the pedal and counter-pedal

$$\frac{U}{\bar{V}} = \frac{\sqrt{\{(P^2 + u)^2(P^2 + v)^2 + P^2P_1^2(P^2 + v)^2 + P^2P_2^2(P^2 + u)^2\}}}{P^3\sqrt{P^2 + P_1^2 + P_2^2}}\,.$$

Calling α_2, β_2, γ_2 and $\overline{\alpha}_2$, $\overline{\beta}_2$, $\overline{\gamma}_2$ the direction-cosines of the normals at corresponding points to the counter-pedal and pedal respectively, this ratio is the value of the determinant

$$\begin{vmatrix} \alpha_2 & \frac{dx}{d\overline{x}} & \frac{dx}{d\overline{y}} & \frac{dx}{d\overline{z}} \\ \beta_2 & \frac{dy}{d\overline{x}} & \frac{dy}{d\overline{y}} & \frac{dy}{d\overline{z}} \\ \gamma_2 & \frac{dz}{d\overline{x}} & \frac{dz}{d\overline{y}} & \frac{dz}{d\overline{z}} \\ & \overline{\alpha}_2 & \overline{\beta}_2 & \overline{\gamma}_2 \end{vmatrix} .$$

Let $F(\xi \eta \zeta) = 0$ represent any surface, and denote by F_ξ, F_η, F_ζ the derivatives of F with respect to ξ, η, ζ. The equation of the tangent plane to this surface is (x', y', z' denoting current coördinates in the plane)

$$(x - \xi) F_\xi + (y - \eta) F_\eta + (z - \zeta) F_\zeta = 0 ;$$

taking the origin as the pole we have for the perpendicular upon the tangent plane

$$p = -\frac{\xi F_\xi + \eta F_\eta + \zeta F_\zeta}{Q} ,$$

where

$$Q^2 = F_\xi^2 + F_\eta^2 + F_\zeta^2 .$$

The direction-cosines of the normal, and therefore of p, are

$$\frac{F_\xi}{Q}, \quad \frac{F_\eta}{Q}, \quad \frac{F_\zeta}{Q} ;$$

and the coördinates of a point on the pedal surface are

$$\overline{x}, \overline{y}, \overline{z} = \frac{pF_\xi}{Q}, \quad \frac{pF_\eta}{Q}, \quad \frac{pF}{Q} .$$

Writing $\frac{p}{Q} = k$ we have for the coördinates of a point on the counter-pedal

$$\begin{aligned} x &= \xi + kF_\xi , \\ y &= \eta + kF_\eta , \\ z &= \zeta + kF_\zeta . \end{aligned}$$

The elimination of ξ, η, ζ between these equations and $F = 0$ will give the equation of the counter-pedal of $F = 0$.

Write the coördinates x, y, z in the form

$$x = \xi + \bar{x}, \quad y = \eta + \bar{y}, \quad z = \zeta + \bar{z}$$

and we have for $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$ the general values

$$\mathfrak{E} = E + \overline{E} + 2\Sigma \frac{d\xi}{du}\frac{d\bar{x}}{du} \text{ \&c.}$$

the quantities $\frac{d\xi}{du}$, $\frac{d\eta}{du}$, $\frac{d\zeta}{du}$ are proportional to the direction-cosines of the tangent to the curve $u = \text{const.}$ traced on the original surface; and $\frac{d\bar{x}}{du}$, $\frac{d\bar{y}}{du}$, $\frac{d\bar{z}}{du}$ are proportional to the direction-cosines of the tangent to the corresponding curve $u = \text{const.}$ traced on the pedal; calling θ the angle between these two tangents the above value of $\mathfrak{E}$ is

$$\mathfrak{E} = E + \overline{E} + 2\sqrt{E\overline{E}}\cos\theta\,.$$

Writing $\theta_1 = (u, \bar{u})$, we may also write

$$\theta_2 = (v, \bar{v}), \quad \phi_1 = (u, \bar{v}), \quad \phi_2 = (\bar{u}, v).$$

The whole set of values is then

$$\mathfrak{E} = E + \overline{E} + 2\sqrt{E\overline{E}}\cos\theta_1,$$

$$\mathfrak{F} = F + \overline{F} + 2\sqrt{E\overline{G}}\cos\phi_1 + 2\sqrt{\overline{E}G}\cos\phi_2,$$

$$\mathfrak{G} = G + \overline{G} + 2\sqrt{G\overline{G}}\cos\theta_2\,.$$

It would be interesting to find the surfaces for which the square of the element of length on the counter-pedal is equal to the sum of the squares of the corresponding elements on the given surface and the pedal, *i. e.* the surfaces which give the relation

$$\mathfrak{E}du^2 + 2\mathfrak{F}dudv + \mathfrak{G}dv^2 = Edu^2 + 2Fdudv + Gdv^2 + \overline{E}du^2 + 2\overline{F}dudv + \overline{G}dv^2$$

or

$$d\sigma^2 = ds^2 + d\bar{\sigma}^2\,.$$

The conditions for this are obviously

$$\cos\theta_1 = 0\,, \quad \cos\theta_2 = 0$$

and

$$\sqrt{E\overline{G}}\cos\phi_1 + \sqrt{\overline{E}G}\cos\phi_2 = 0\,;$$

or otherwise

$$\Sigma \frac{d\xi}{du} \frac{d\bar{x}}{du} = 0 , \quad \Sigma \frac{d\xi}{dv} \frac{d\bar{x}}{dv} = 0$$

$$\Sigma \frac{d\xi}{du} \frac{d\bar{x}}{dv} + \Sigma \frac{d\xi}{dv} \frac{d\bar{x}}{du} = 0 .$$

The quantities $\forall$, ꓭ, Ɔ may be written in the form

$$\forall = A + \bar{A} - \left\{ \frac{d(\eta, \bar{z})}{d(u, v)} - \frac{d(\zeta, \bar{y})}{d(u, v)} \right\}$$

$$\text{ꓭ} = B + \bar{B} - \left\{ \frac{d(\zeta, \bar{x})}{d(u, v)} - \frac{d(\xi, \bar{z})}{d(u, v)} \right\}$$

$$\text{Ɔ} = C + \bar{C} - \left\{ \frac{d(\xi, \bar{y})}{d(u, v)} - \frac{d(\eta, \bar{x})}{d(u, v)} \right\} .$$

There are many other general formulas that one might write down, but it is hardly worth while to do so.